# MATH 320 NOTES, WEEK 2

Recall: Let $V$ be a vector space, $W \subset V$ is a subspace iff $\vec{0} \in W$, and $W$ is closed under vector addition and multiplication.

Some trivial examples: both $\{\vec{0}\}$ and $V$ are subspaces of $V$.

Let us recall some more set theoretic notation:

- Intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- Union: $A \cap B = \{x \mid x \in A \text{ or } x \in B\}$
- Set difference: $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$

Recall the key theorem that $W$ is a subspace of $V$ iff $\vec{0} \in W$, and $W$ is closed under both vector addition and scalar multiplication. We can actually show the last two at once:

**Corollary 1.** *$W$ is a subspace of a vector space $V$ over $F$ iff $\vec{0} \in W$, and for all $x, y \in W$ and $c \in F$, we have $cx + y \in W$.*

*Proof.* Exercise.

□

**Definition 2.** *A matrix $A \in M_{n,n}(F)$ is called skew-symmetric iff $A^t = -A$.*

**Example.** Show that the set of all skew-symmetric $n$ by $n$ matrices is a subspace of $M_{n,n}(F)$.

**Theorem 3.** *Suppose that $W_1$ and $W_2$ are two subspaces of a vector space $V$. Then their intersection $W_1 \cap W_2$ is also a subspace.*

*Proof.* We have to check the three requirements of being a subspace hold for $W_1 \cap W_2$.

(1) $\vec{0} \in W_1$ and $\vec{0} \in W_2$, so $\vec{0} \in W_1 \cap W_2$.
(2) Suppose $x, y \in W_1 \cap W_2$. Since $W_1$ is a subspace and $x, y \in W_1$, we have $x + y \in W_1$. Also, since $W_2$ is a subspace and $x, y \in W_2$, we have $x + y \in W_2$. So $x + y \in W_1 \cap W_2$.
(3) Suppose $x \in W_1 \cap W_2, c \in F$. Since $W_1$ is a subspace and $x \in W_1$, we have $cx \in W_1$. Also, since $W_2$ is a subspace and $x \in W_2$, we have $cx \in W_2$. So $cx \in W_1 \cap W_2$.

□

*Remark* 4. Similarly, any intersection of subspaces $W_1 \cap W_2 ... \cap W_n$ is a subspace.

**Question:** What about $W_1 \cup W_2$?

This question motivates the following definition:

**Definition 5.** *Let $W_1$ and $W_2$ be two subspaces of $V$. Define the sum $W_1 + W_2 = \{x + y \mid x \in W_1, y \in W_2\}$.*

**Lemma 6.** *If $W_1$ and $W_2$ are two subspaces of $V$, then $W_1 + W_2$ is also a subspace.*

*Proof.* $\vec{0} = \vec{0} + \vec{0} \in W_1 + W_2$.

Suppose that $x \in W_1 + W_2, y \in W_1 + W_2$ and $c$ is a scalar. Then

- $x = x_1 + x_2$ for some $x_1 \in W_1$ and $x_2 \in W_2$ and
- $y = y_1 + y_2$ for some $y_1 \in W_1$ and $y_2 \in W_2$.

So, $cx+y = c(x_1+x_2)+(y_1+y_2) = cx_1+cx_2+y_1+y_2 = (cx_1+y_1)+(cx_2+y_2) \in W_1 + W_2$.

□

**Definition 7.** *Let $W_1$, $W_2$, and $V$ be vector spaces. We say that*

$$V = W_1 \oplus W_2$$

*iff*

(1) *$V = W_1 + W_2$ and*
(2) *$W_1 \cap W_2 = \{\vec{0}\}$.*

*We say that $W_1 \oplus W_2$ is the* **direct sum** *of $W_1$ and $W_2$.*

**Exercise:** Let $W_1$ be the space of all symmetric matrices in $M_{2,2}(F)$, and let $W_1$ be the space of all skew-symmetric matrices in $M_{2,2}(F)$. Suppose also that the characteristic of $F$ is not two. Show that $M_{2,2}(F) = W_1 \oplus W_2$.

*Proof.* First we show that $M_{2,2}(F) = W_1 + W_2$. Clearly $W_1 + W_2 \subset M_{2,2}(F)$. Now, suppose that $A \in M_{2,2}(F)$. Say $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $A = \begin{pmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{pmatrix} + \begin{pmatrix} 0 & \frac{b-c}{2} \\ \frac{c-b}{2} & 0 \end{pmatrix} \in W_1 + W_2$.

Next we have to show that $W_1 \cap W_2 = \{\vec{0}\}$. Suppose that $A \in W_1 \cap W_2$. Then $A = A^t = -A$. So for any entry of $A$, $a_{ij} = -a_{ij} = 0$, since the characteristic of $F$ is not 2. Then $A = O$.

□

**Exercise:** Let $V$ be a vector space, and $x, y \in V$.

(1) Show that $\{ax \mid a \in F\}$ is a subspace of $V$.
(2) Show that $\{ax + by \mid a, b \in F\}$ is a subspace of $V$.

**Section 1.4 Linear Combinations**

We start with the key notion:

**Definition 8.** *Suppose that $V$ is a vector space over $F$ and $S \subset V$ is nonempty. A vector $x \in V$ is a* **linear combination** *of vectors in $S$, if*

$$x = a_1 v_1 + a_2 v_2 + ... + a_n v_n$$

*for some vectors $v_1, ..., v_n$ in $S$ and scalars $a_1, ..., a_n$ in $F$.*
*The* **span of** *$S$, $Span(S)$ is the set of all linear combinations of vectors in $S$.*

Also define $Span(\emptyset) = \{\vec{0}\}$.

Examples. Consider $\mathbb{R}^2$.

(1) $Span(\{\langle 1,0\rangle, \langle 0,1\rangle\}) = \mathbb{R}^2$;
(2) $Span(\langle 1,0\rangle) = \{\langle a,0\rangle \mid a \in \mathbb{R}\}$;
(3) $Span(\langle 1,1\rangle) = \{\langle a,a\rangle \mid a \in \mathbb{R}\}$;
(4) $Span(\langle 17,17\rangle) = Span(\langle 1,1\rangle)$;
(5) $Span(\{\langle 1,1\rangle, \langle 0,2\rangle\}) = \mathbb{R}^2$;

For a vector space $V$, if $V = Span(S)$, we say that $S$ **generates** $V$ or that $S$ **spans** $V$.

**Theorem 9.** *Let $V$ be a vector space and $S \subset V$. Then $Span(S)$ is a subspace. Moreover, $Span(S)$ is the smallest subspace containing $S$ i.e. if $W$ is a subspace of $V$ with $S \subset W$, then $Span(S) \subset W$.*

*Proof.* Let us first check that $Span(S)$ is a subspace, by verifying that the three requirements for a subspace hold. If $S = \emptyset$, then $Span(S) = \{\vec{0}\}$, which is clearly a subspace. So, assume that $S$ is nonempty.

(1) Using that $S$ is nonempty, pick any vector $x \in S$. Then

$$\vec{0} = 0x \in Span(S).$$

(2) For closure under vector addition, suppose that $x, y \in Span(S)$. Then for some vectors $v_1, ..., v_n, w_1, ..., w_k$ in $S$ and scalars $a_1, ...., a_n, b_1, ..., b_k$, we have that

$$x = a_1 v_1 + a_2 v_2 + ... + a_n v_n, \text{ and } y = b_1 w_1 + b_2 w_2 + ... + b_k w_k.$$

Then

$$x + y = a_1 v_1 + a_2 v_2 + ... + a_n v_n + b_1 w_1 + b_2 w_2 + ... + b_k w_k \in Span(S).$$

(3) For closure under scalar multiplication, suppose that $x \in Span(S)$ and $c$ is a scalar. Then for some vectors $v_1, ..., v_n$ in $S$ and scalars $a_1, ...., a_n$, we have that

$$x = a_1 v_1 + a_2 v_2 + ... + a_n v_n.$$

Then $cx = ca_1 v_1 + ca_2 v_2 + ... + ca_n v_n \in Span(S)$.

It follows that $Span(S)$ is a subspace of $V$.

Now, for the second part of the theorem suppose that $W$ is a subspace of $V$ and $S \subset W$. We have to show that $Span(S) \subset W$.

Let $x \in Span(S)$. Then $x = a_1v_1 + a_2v_2 + ... + a_nv_n$, where $v_1, ..., v_n$ are vectors in $S$ and $a_1, ..., a_n$ are scalars. By closure under scalar multiplication $a_iv_i \in W$, for each $i \leq n$. And by closure under vector addition the sum $x \in W$. $\square$

Example. $\mathbb{R}^3$.

- The vectors $(1, 0, 0), (0, 1, 0)$, and $(0, 0, 1)$ span $\mathbb{R}^3$.
- The vectors $(1, 1, 1), (0, -2, 0)$, and $(0, 1, 1)$ span $\mathbb{R}^3$.

Example. Polynomials

- The polynomials $x^2, x$, and 1 span $P_2(\mathbb{R})$.
- The polynomials $x^2 + x + 1, 5x^2$ and 1 span $P_2(\mathbb{R})$.
- The polynomials $x + 1$ and $2x$ span $\mathbb{P}_1(\mathbb{R})$. Note that this is a subspace of $\mathbb{P}_2(\mathbb{R})$.
- The polynomials $1, x, x^2, x^3, ..., x^n, ....$ span $P(\mathbb{R})$.

Example. Matrices

- The matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ span $\mathbb{M}_{2,2}(\mathbb{R})$.
- The matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ span the space of all symmetric matrices in $\mathbb{M}_{2,2}(\mathbb{R})$.
- The matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ span the space of all diagonal matrices in $\mathbb{M}_{2,2}(\mathbb{R})$.